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Arbitrage and Existence of Equilibrium in Infinite Asset Markets

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This paper develops a framework for a general equilibrium analysis of asset markets when the number of assets is infinite. Such markets have been studied in the context of asset pricing theories. Our main results concern the existence of an equilibrium. We show that an equilibrium exists if there is a price system under which no investor has an arbitrage opportunity. A similar result has been previously known to hold in finite asset markets. Our extension to infinite assets involves a concept of an arbitrage opportunity which is different from the one used in finite markets. An arbitrage opportunity in finite asset markets is a portfolio that guarantees non-negative payoff in every event, positive payoff in some event, and has zero price. For the case of infinite asset markets, we introduce a concept of sequential arbitrage opportunity which is a sequence of portfolios which increases an investor's utility indefinitely and has zero price in the limit. We show that a sequential arbitrage opportunity and an arbitrage portfolio are equivalent concepts in finite markets but not in their infinite counterpart.

1. INTRODUCTION

Modern asset pricing theories study pricing relations arising in models of competitive asset markets. The classical Capital Asset Pricing Model of Lintner (1965) and Sharpe (1964) is an example of such a theory which derives sharp predictions about asset prices from a simple equilibrium model of asset trading. The critical assumption of the CAPM is that investors are guided in their investment decisions only by the mean and the variance of a payoff of a portfolio. An alternative asset pricing theory is the Arbitrage Pricing Theory of Ross (1976). The APT derives an (approximate) pricing relation in the limit as the number of traded assets increases indefinitely. The critical assumptions of the APT are the factor structure of asset payoffs and the absence of (approximate) arbitrage opportunities. The CAPM, and—more generally—a finite asset market model is well-understood from the point of view of the general equilibrium theory, and conditions guaranteeing the existence of an equilibrium are well-known (see Hart (1974), Hammond (1983), Nielsen (1989, 1990), Page (1987)). In contrast, the APT is in its standard derivation a partial equilibrium model with prices exogenously given (see Chamberlain and Rothschild (1983), and Chamberlain (1983) for the most comprehensive study). A general equilibrium analyses of the APT requires a countably infinite number of assets, optimizing investors, and an endogenous determination of equilibrium prices. This paper develops a framework for such an analysis.

The prototypical equilibrium model of finite asset markets which includes the CAPM as a special case is due to Hart (1974). In Hart's model assets are described by their end-of-period (random) payoffs. Investors trade assets at the beginning of a time period so as to maximize expected utility of a payoff of a portfolio subject to a budget constraint. They may have diverse expectations about asset payoffs. Hart's model has the same structure as the standard Arrow–Debreu model with the only difference that agents (investors) choose portfolios instead of commodity bundles. This difference has, however, profound implications for the problem of the existence of an equilibrium. Since short-sales of assets are permitted, sets of feasible portfolios are, in general, not bounded below. This is a consequence of the fact that typically there are portfolios with negative holdings of some assets that have positive payoffs with (subjective) probability one. An arbitrary replication of such a portfolio is feasible. It is worth pointing out that feasible portfolio set is not the entire portfolio space, if an investor's end-of-period wealth is restricted to be non-negative. A condition that guarantees the existence of an equilibrium in a finite asset market economy is that the economy is arbitrage-free (see Werner (1987), and Nielsen (1989); for a characterization of arbitrage-free economies in terms of a condition of overlapping expectations see Hammond (1983)). An economy is arbitrage-free if there is a price system under which no investor has an arbitrage portfolio. An arbitrage portfolio is a portfolio that guarantees non-negative payoff in every event, positive payoff in some event of positive probability, and has zero or negative price.

The purpose of this paper is to extend the existence of equilibrium results to asset markets with infinitely many assets. More specifically, we extend the principle of the existence of an equilibrium in arbitrage-free economies to infinite asset markets. Our results require, however, a modification of the notion of an arbitrage opportunity. It has long been recognized in the literature on asset markets that the concept of the absence of an arbitrage opportunity as developed for finite markets is far too weak for infinite markets (see Kreps (1981)). We provide in Section 4 a detailed discussion of concepts of arbitrage. The need for a modified notion of an arbitrage opportunity in infinite markets can be loosely explained as follows: If there is an arbitrage portfolio (with non-negative payoff in every event, positive payoff in some event of positive probability, and zero or negative price), then an investor would keep increasing without a limit the amount of this portfolio she holds. This would result in an unbounded sequence of portfolios increasing her utility while being budget feasible. In finite markets whenever there is an unbounded sequence of budget feasible portfolios increasing the (expected) utility, then there must be an arbitrage portfolio (see Proposition 1 in Section 4.1). In this sense arbitrage portfolios fully characterize unbounded sequences of portfolios that increase an investor's utility while being budget feasible. This logic breaks down in the infinite-dimensional case. In Section 4 we provide an example of an investor and a price system such that there is no arbitrage portfolio but there is a way of increasing the investor's utility without limit at no cost. Therefore, an arbitrage opportunity in infinite markets has to be defined explicitly as a sequence of portfolios rather than a single portfolio in order to characterize opportunities of increasing an investor's utility at zero cost.

A sequence of portfolios increasing an investor's utility indefinitely and such that the market values of the portfolios converge to zero will be called a *sequential arbitrage opportunity*. This concept is similar to the notion of approximate arbitrage in the APT (see Ross (1976)), which is a sequence of portfolios such that the expected values of the payoffs converge to infinity, the variances of the payoffs converge to zero, and the market values of the portfolios converge to zero. It is, however, much weaker since it bears no relation to a risk-free payoff in the limit and is utility-dependent.

A price system is arbitrage-free if no investor in the market has a sequential arbitrage opportunity, and an economy is arbitrage-free if the set of arbitrage-free prices is non-empty. Our main result establishes the existence of an equilibrium for an infinite-dimensional arbitrage-free economy.

The model of this paper is an extension of the one studied in Werner (1987). It is a standard general equilibrium model with an infinite-dimensional commodity space with the notable distinction that agents' choice sets are not assumed to be bounded below. This distinction has two important implications: First, the existing existence of equilibrium theorems for infinite-dimensional commodity spaces (see Aliprantis, Brown and Burkinshaw (1989), and Mas-Colell and Zame (1991) for surveys of these results) cannot be applied. Second, it makes our analysis applicable to asset markets models. We find it appropriate to study the existence of equilibrium problem in a general setting in order to separate complications caused by the absence of the assumption of bounded-below choice sets from specific features of asset trading models. In our abstract model we use the terminology of the general equilibrium theory such as a "commodity" and a "consumption set." Nevertheless, a reader should most naturally have an asset market interpretation in mind, and thus think about a "commodity bundle" as a portfolio of assets (i.e. a list of shareholdings of all assets), or a combination of a portfolio and a commodity bundle for current consumption. Section 6 provides an example of an infinite asset market model as a special case of the general model underlying the rest of the paper, and can be consulted for details of the suggested interpretation.

The model is presented in Section 2. Section 3 contains an existence of equilibrium result for an economy with arbitrary consumption sets. In Section 4 we introduce the concept of a sequential arbitrage opportunity and examine its relationship with alternative concepts. In Section 5 we show that an equilibrium price system is arbitrage-free, and we derive our main existence of equilibrium result for an arbitrage-free economy.

Equilibrium models related to the model of this paper have been studied by Cheng (1991), without any reference to arbitrage, and by Chichilnisky and Heal (1993*a*) under the restriction that the commodity space is a Sobolev space. In place of our condition of an economy being arbitrage-free, Chichilnisky and Heal impose a condition of the absence of unbounded mutually beneficial trades among the agents.

2. THE MODEL

We shall consider an exchange economy with a commodity space E. The space E is assumed to be a locally convex, topological vector space with topology τ. There are m consumers indexed by $i=1,\ldots,m$. Each consumer i is described by a consumption set $X_i \subset E$, and an initial endowment $e_i \in X_i$. The preferences of consumer i are represented by a utility function $u_i: X_i \to R$. The basic assumptions about consumers' characteristics that will be maintained throughout the paper are the following:

$(A1)$ X_i is closed, and convex.

$(A2)$ u_i is τ-continuous, and there is $v_i \in E$ such that $u_i(x+\alpha v_i) > u_i(x)$ for every $x \in X_i$, and $\alpha > 0$.

It should be emphasized that we do not assume that the commodity space is a Riesz space or that consumption sets are the positive cone. The latter is of special importance for models of asset markets, where a commodity is a share of an asset.

We shall refer to a tuple $(X_i, u_i, e_i)_{i=1,\dots,m}$ as an (*exchange*) *economy*. If $E=R^{\ell}$ for some ℓ, an economy will be called finite-dimensional. Otherwise, it is infinite-dimensional—the case of interest.

The space of continuous linear functionals on E will be denoted by E'. E' constitutes the price space for our model with a generic element $p \in E'$ being a price system.

3. EQUILIBRIUM

Any m-tuple of consumption plans $(x_1, \dots, x_m)$ such that $x_i \in X_i$ will be called an *allocation*. If $\sum_{i=1}^{m} x_i = e$, where $e=\sum_{i=1}^{m} e_i$ is the total endowment, then the allocation is *attainable*. Let A denote the set of all attainable allocations, and let $U=\{u=(u_1, \dots, u_m) \in \mathbb{R}^m : u_i(e_i) \leqq u_i \leqq u_i(x_i),\ i=1, \dots, m$ for some $(x_1, \dots, x_m) \in A\}$ be the set of individually rational attainable utility levels (*utility set*, for short).

A *competitive equilibrium* is an attainable allocation $(x_1, \dots, x_m) \in A$ and a non-zero price $p \in E'$ such that $x_i \in B_i(p)$ and $u_i(x_i) \geqq u_i(x)$ for every $x \in B_i(p)$, where $B_i(p) = \{x \in X_i : px \leqq pe_i\}$ is the budget set.

The existence of equilibrium theorem requires three more assumptions in addition to assumptions A1 and A2. The first of these assumptions is standard.

(A3) u_i is quasi-concave.

The second is not unusual for equilibrium theory in infinite-dimensional spaces (see Aliprantis, Brown and Burkinshaw (1989), Mas-Colell and Zame (1991)). In Section 5 we discuss a relationship between this condition and a condition of the absence of arbitrage opportunities.

(A4) The utility set U is compact.

Thirdly, we impose a condition that guarantees that preferred sets are price supported (i.e. for every $x \in X_i$, if $u_i(x') \geqq u_i(x)$, then $px' \geqq px$ for some $p \in E'$). Let $P_i(x)$ denote the preferred-to-x set, i.e. $P_i(x) = \{x' \in X_i : u_i(x') \geqq u_i(x)\}$, for $x \in X_i$.

(A5) int $P_i(x) \neq \emptyset$ for every $x \in X_i$.

We are now in a position to state our main existence theorem. The theorem establishes the existence of a quasi-equilibrium. A *quasi-equilibrium* is an attainable allocation $(x_i)_{i=1}^{m} \in A$, and a non-zero price $p \in E'$ such that $px \geqq pe_i$ for every $x \in X_i$ with $u_i(x) \geqq u_i(x_i)$. A quasi-equilibrium $((x_i)_{i=1}^{m}, p)$ such that $px_i > \min pX_i$ for every i, is an equilibrium. Conditions to assure the minimum wealth constraint are standard. An important example is the condition $e_i - \varepsilon v_i \in X_i$, for some $\varepsilon > 0$.

Theorem 1. *If an economy satisfies Assumptions* $A1$, $A2$, $A3$, $A4$, *and* $A5$ *for every* $i=1, \dots, m$, *then it has a quasi-equilibrium.*

The proof can be found in the Appendix. The basic argument is that of Negishi which was extended to infinite-dimensional economies by Bewley (1969), Magill (1981), and Mas-Colell (1986).

We emphasize that the assumptions of our theorem do not require utility functions to be monotonic, or the consumption sets to be bounded below. Condition A5 implies that consumption set X_i has non-empty interior ruling out the positive cone in many spaces as a possible consumption set. The positive cone is, however, not a typical choice set in asset market models. If the consumption set has non-empty interior, e.g., the consumption set is the entire commodity space, and the utility function is continuous (A2), then A5 holds. The only role of Assumption A5 is to assure the price supportability of preferred sets (both for an individual consumer and for the whole economy), and could be replaced by any other condition sufficient for that (e.g. uniform properness when consumption sets are the positive cone of a Riesz commodity space). In this sense our result generalizes Theorem 7.1 in Mas-Colell and Zame (1991).

4. ARBITRAGE

This section is devoted to a discussion of concepts of an arbitrage opportunity. The first concept, which we call a free lunch, is an extension of the standard concept of an arbitrage portfolio in finite asset markets. We shall argue that it is inadequate for the purpose of an equilibrium analysis of infinite markets. We introduce an alternative concept and investigate its properties.

Let C be a closed and convex subset of E. The recession (asymptotic) cone of C is the set of all vectors $\hat{x} \in E$ such that $x + \lambda\hat{x} \in C$ for every $x \in C$ and every $\lambda \geqq 0$. The recession cone of C, denoted by AC, is closed and convex. We show in Appendix that $AC = \{\hat{x} \in E: \hat{x} = \lim \lambda_n x_n$ for some sequences $\{x_n\} \subset C$ and $\{\lambda_n\} \subset R_+$ with $\lim \lambda_n = 0\}$. An element of AC is called a direction of recession of C. If $\hat{x} \in AC$ and $-\hat{x} \in AC$, then $\hat{x}$ is a direction in which C is linear.

Consider consumer i with utility function u_i on X_i. We shall strengthen assumption A3 to:

($A3'$) u_i is concave.[1]

A direction $\hat{x}$ is which X_i is linear and such that $u_i(x + \lambda\hat{x}) = u_i(x)$ for every $\lambda \in R$ and every $x \in X_i$ will be called a direction in which u_i is constant. A commodity bundle $\hat{x} \in AX_i$ such that $u_i(x + \hat{x}) \geqq u_i(x)$ for every $x \in X_i$ and u_i is not constant in the direction $\hat{x}$ will be called a *useful* commodity bundle for utility u_i. One can show that a commodity bundle is useful for concave utility function u_i, if and only if it is a direction of recession of the preferred set $P_i(x)$ for every $x \in X_i$ but not a direction in which $P_i(x)$ is linear.

Let $p \in E'$ be a price system.

Definition 1. A free lunch for consumer i (with respect to p) is a commodity bundle $\hat{x} \in E$ such that $p\hat{x} \leqq 0$, and $\hat{x}$ is useful for utility u_i.

This concept of a free lunch was introduced in Werner (1987) (under the name of arbitrage opportunity). In the context of financial asset markets, where a commodity

1. The assumption of concavity greatly facilitates the analysis of arbitrage opportunities. It is satisfied in standard models of asset markets where the utility of a portfolio is given by a concave expected utility of its payoff (see Section 6).

bundle is a portfolio of assets, and the utility of a portfolio is the expected utility of its payoff, a free lunch is a portfolio with a non-negative payoff with probability one, positive payoff with positive probability, and zero or negative value (see Section 6). In this sense the concept of free lunch is a natural extension of the concept of arbitrage portfolio in asset markets.

Kreps (1981) pointed out that many consequences of the condition of the absence of free lunch do not extend to infinite-dimensional economies. In accordance with this observation, concepts of an arbitrage opportunity used in the literature on infinite asset markets are different. For instance, in the context of the APT (see Ross (1976)) it is a sequence of portfolios with the expected values of the payoffs converging to infinity, the variances of the payoffs converging to zero, and the market values of the portfolios converging to zero.

In our context an arbitrage opportunity is defined as follows: Let $\bar{u}_i = \sup_{x \in X_i} u_i(x)$ ($\bar{u}_i$ can be finite or $+\infty$).

Definition 2. A sequential arbitrage opportunity for consumer i (with respect to p) is a sequence of commodity bundles $\{\hat{x}_n\} \subset E$ such that $e_i + \hat{x}_n \in X_i$, $\lim u_i(e_i + \hat{x}_n) = \bar{u}_i$, and $\lim p\hat{x}_n \leq 0$.

We shall call a price system *arbitrage-free* for consumer i if there is no sequential arbitrage opportunity for i.

4.1. *Free lunch versus sequential arbitrage opportunity.*

In general neither the existence of a free lunch implies the existence of a sequential arbitrage opportunity nor the converse. However, in a finite-dimensional economy we have:

Proposition 1. *Suppose* $A1$, $A2$, *and* $A3'$ *hold, and* $E = R^\ell$. *If* $p \in R^\ell$ *admits no free lunch for consumer i, then it is arbitrage-free for i.*

Proof. We first consider the case when there is no direction in which u_i is constant. Suppose that there is a sequential arbitrage opportunity $\{\hat{x}_n\}$ for consumer i at p. Clearly $\{\hat{x}_n\}$ is unbounded. Let $\hat{x}$ by any cluster point of $\{\hat{x}_n / \|x_n\|\}$. We have $\hat{x} \neq 0, p\hat{x} \leq 0$ and $\hat{x}$ is a direction of recession of $P_i(x)$ for every $x \in X_i$. Thus $\hat{x}$ is a free lunch which contradicts the assumption.

The proof in the case when there are directions in which u_i is constant proceeds by restricting the utility function and prices to the subspace orthogonal to directions in which u_i is constant, and applying the argument above. Details are omitted. ||

Under an additional condition on the utility function a result converse to Proposition 1 holds regardless whether the commodity space is finite- or infinite-dimensional.

Proposition 2. *Suppose that* $\lim u_i(e_i + n\hat{x}) = \bar{u}_i$ *for every useful commodity bundle* $\hat{x}$. *Then every arbitrage-free price system admits no free lunch.*

Proof. Straightforward. ||

Thus in a finite-dimensional economy, if the utility function satisfies the assumption of Proposition 2, then a price system is arbitrage-free if and only if it admits no free lunch.

If the assumption of Proposition 2 is not satisfied, then the condition of being arbitrage-free is weaker. The following simple example demonstrates that the assumption of Proposition 2 is indispensable:

Example 1. Let $E=R^2$, $u(x_1, x_2)=\min\{x_1, x_2\}$, and $e=(2, 0)$. The price system $p=(1, 0)$ is arbitrage-free but it admits a free lunch being $\hat{x}=(0, 1)$. Note that $\hat{x}$ is useful but $\lim u(e+n\hat{x})=2<\bar{u}=+\infty$.

Proposition 1 does not generalize to infinite-dimensional commodity spaces. In such spaces 0 may be a cluster point of the sequence $\{\hat{x}_n/\|x_n\|\}$ which invalidates our proof. The following example shows a price system under which there is no free lunch for a consumer in the commodity space ℓ_∞, but there is a sequential arbitrage opportunity.

Example 2. Let $E=\ell_\infty$ and $E'=\ell_1$. Consider the utility function $u:\ell_\infty^+\to R$ defined by $u(x)=\sum_{n=1}^{\infty}\delta^n(x_n)^{\frac{1}{2}}$ where $x=(x_1, x_2, \ldots)$, and $0<\delta<1$. Let $e=0$ be the initial endowment. Every commodity bundle $\bar{x}\in\ell_\infty^+$, $\bar{x}\neq 0$, is useful, and therefore every strictly positive price system admits no free lunch. Let us consider a price system $p=(p_1, p_2, \ldots)\in\ell_1$ given by $p_n=\delta^{4n}$. We claim that p is not arbitrage-free. Let $x^k\in\ell_\infty^+$ be defined by $x_n^k=\delta^{-3n}$ for $n=k$ and zero otherwise, $k=1, 2, \ldots$. We have $u(e+x^k)=\delta^k\delta^{-\frac{3}{2}k}=\delta^{-\frac{1}{2}k}\to+\infty$. On the other hand $px^k=\delta^k\to 0$. Thus $\{x^k\}$ is a sequential arbitrage opportunity with respect to p.

The proof of Proposition 1 and Example 2 illustrate why the concept of free lunch is in general not adequate for infinite markets. In infinite markets (unlike in their finite counterpart) an unbounded sequence of consumptions that increases utility may not have a corresponding useful commodity bundle.

5. ARBITRAGE AND EQUILIBRIUM

One of the main issues we address in this paper is a relationship between an equilibrium and the absence of sequential arbitrage opportunities. In a finite-dimensional economy every equilibrium price system does not admit a free lunch (provided that utility functions have no half-lines in indifference sets). Moreover, the existence of a price system which does not admit a free lunch is sufficient for the existence of an equilibrium (see Proposition 2 (ii), and Theorem 1 in Werner (1987)). In this section we investigate analogous results for an infinite-dimensional economy using the concept of sequential arbitrage opportunity.

We call a price system *viable* for consumer i, if the demand of consumer i is well-defined, i.e. there is $x_i\in B_i(p)$ such that $u_i(x_i)\geqq u_i(x)$ for every $x\in B_i(p)$.

Proposition 3. *Suppose A1, A2 and A3′ hold. If $p\in E'$ is viable for consumer i, and $pe_i>\min pX_i$, then p is arbitrage-free for consumer i.*

Proof. Suppose the contrary. Then there is a sequence $\{x_n\}$ such that $\lim u_i(e_i+x_n)=\bar{u}_i$, and $\lim p(e_i+x_n)\leqq pe_i$. Let $\bar{x}\in X_i$ be such that $p\bar{x}<pe_i$. For $0\leqq\lambda\leqq 1$, $\lambda(e_i+x_n)+(1-\lambda)\bar{x}\in X_i$. Let M be such that $u_i(x_i)<M<\bar{u}_i$. Since u_i is concave, there exists $0<\lambda_0<1$ such that $\liminf u_i(\lambda_0(e_i+x_n)+(1-\lambda_0)\bar{x})\geqq M$. For n large enough, we have $u_i(\lambda_0(e_i+x_n)+(1-\lambda_0)\bar{x})>u_i(x_i)$ and $p(\lambda_0(e_i+x_n)+(1-\lambda_0)\bar{x})<pe_i$ which contradicts the optimality of x_i in the budget set $B_i(p)$. ||

An immediate corollary is the following:

Theorem 2. *If p is an equilibrium price system, and $pe_i > \min pX_i$ for every $i = 1, \ldots, m$, then p is arbitrage-free.*

In the remainder of this section we investigate the sufficiency of the condition that there is a price system which is arbitrage-free for every consumer for the existence of an equilibrium. To facilitate the discussion we shall introduce the following:

Definition 3. An economy is *arbitrage-free* if there exists a price system which is arbitrage-free for every consumer.

We shall focus our attention on Assumption A4 of compactness of the utility set. The utility set is compact if and only if it is closed and bounded. In economies with consumption sets being the positive cone boundedness of the utility set is a consequence of (order) boundedness of the set of attainable allocations. In our case boundedness of the utility set is a legitimate concern, if some utility functions are unbounded from above. Unbounded from above utility functions are frequently used in finance (e.g. constant relative risk aversion utility functions). Proposition 4 shows that the utility set of an arbitrage-free economy is bounded regardless of whether utility functions are bounded from above or not.

Proposition 4. *Suppose A1, A2 and A3′ hold for every $i = 1, \ldots, m$. If the economy is arbitrage-free, then the utility set is bounded.*

The proof of Proposition 4 consists of three steps. The first two steps are Lemmas 1 and 2 for which A1, A2, and A3′ are assumed to hold for every i.

Lemma 1. *If p is arbitrage-free for consumer i, then $px > b$ for some b and every $x \in P_i(e_i)$.*

Proof. Suppose the contrary. Then there exists a sequence $\{x_n\} \subset E$ such that $e_i + x_n \in P_i(e_i)$ for every n, and $\lim px_n = -\infty$. Let $\alpha_n = -px_n$. For each n, define the set $W_n = \{x \in P_i(e_i) : px \leqq \sqrt{\alpha_n}\}$. We have $P_i(e_i) \subset \bigcup_{n=1}^{\infty} W_n$. Let $\{z_n\}$ be a sequence such that $z_n \in W_n$ and $\lim u_i(e_i + z_n) = \bar{u}_i$. Consider a sequence $\{y_n\}$ defined by $y_n = \lambda_n x_n + (1 - \lambda_n) z_n$, where $\lambda_n = 1/(1 + \sqrt{\alpha_n})$. We have $py_n = \lambda_n px_n + (1 - \lambda_n) pz_n \leqslant \lambda_n(-\alpha_n) + (1 - \lambda_n) \sqrt{\alpha_n} = 0$. Moreover, $u_i(e_i + y_n) \geqq \lambda_n u_i(e_i + x_n) + (1 - \lambda_n) u_i(e_i + z_n)$. Since $\lambda_n \to 0$, we obtain $\lim u_i(e_i + y_n) = \bar{u}_i$, and $\{y_n\}$ is a sequential arbitrage opportunity. This is a contradiction. ||

Lemma 2. *Suppose that the utility function u_i is unbounded, i.e. $\bar{u}_i = +\infty$. If p is arbitrage-free for consumer i, then* $\lim px_n = +\infty$ *for every sequence of consumption plans $\{x_n\} \subset X_i$ such that* $\lim u_i(x_n) = +\infty$.

Proof. Let $\{x_n\} \subset X_i$ be a sequence such that $\lim u_i(x_n) = +\infty$ and $\lim px_n < +\infty$. Let $\hat{x}_n = x_n - e_i$, and $\alpha = \lim p\hat{x}_n$. We have $\alpha < \infty$. There is a sequence $\{\gamma_n\} \subset \mathbb{R}_+$ such that

$\lim u_i(e_i+\gamma_n\hat{x}_n)=+\infty$ and $\gamma_n \to 0$. Indeed, let $\gamma_n=1/\sqrt{u_i(x_n)}$. Then, by concavity of u_i,

$$u_i(e_i+\gamma_n\hat{x}_n)=u_i(\gamma_n(e_i+\hat{x}_n)+(1-\gamma_n)e_i)\geqq \gamma_n u_i(x_n)+(1-\gamma_n)u_i(e_i)$$
$$=\frac{1}{\gamma_n}+(1-\gamma_n)u_i(e_i).$$

Therefore $\lim u_i(e_i+\gamma_n\hat{x}_n)=+\infty$. Since $\lim p(\gamma_n\hat{x}_n)=0$, $\{\gamma_n\hat{x}_n\}$ is a sequential arbitrage opportunity contradicting our assumption. ‖

We are now in a position to prove Proposition 4.

Proof. Suppose by contrary that U is unbounded. Then there exists a sequence $\{u^n\}\subset U$ such that $\lim u^n_{i_0}=+\infty$ for some i_0. Let $x^n_i \in P_i(e_i)$ be such that $u_i(x^n_i)\geqq u^n_i$ and $\sum_{i=1}^m x^n_i=e$ for every n, and $i=1,\ldots,m$. We have $\lim u_{i_0}(x^n_{i_0})=+\infty$ and therefore (Lemma 2) $\lim px^n_{i_0}=+\infty$ for an arbitrage-free price system p. By Lemma 1, $\liminf px^n_i>-\infty$ for every i. Thus, we obtain a contradiction to $p\sum_{i=1}^m x^n_i=pe<+\infty$. ‖

If the assumptions of Proposition 4 are satisfied, the economy is arbitrage-free, and the utility set is closed, then condition A4 holds. Thus, we have the following existence of equilibrium result for an arbitrage-free economy, as an immediate corollary to Theorem 1 and Proposition 4:

Theorem 3. *If an economy is arbitrage-free, satisfies Assumptions A1, A2, A3′ and A5 for every $i=1,\ldots,m$, and has closed utility set, then it has a quasi-equilibrium.*

Closedness of the utility set is a frequent assumption in equilibrium theory with infinite-dimensional commodity spaces. It is independent of the other assumptions of Theorem 3, in particular of the assumption that the economy is arbitrage-free. In the following example the economy satisfies conditions A1, A2, A3′ and A5, and is arbitrage-free (and therefore has bounded utility set), but the utility set is not closed. Moreover, there is no (quasi-) equilibrium.

Example 3. (Aliprantis, Brown and Burkinshaw (1989), Example 3.3.7). Let the commodity space be the space of continuous functions on the interval [0, 1] with the sup norm, and let the price space be the space of measures on [0, 1]. Thus $E=C[0,1]$, and $E'=ca[0,1]$. There are two consumers, $i=1,2$, each with the consumption set being the positive cone E_+, and the endowment $e_i \in E_+$ given by $e_i(t)=1$ for all $t\in[0,1]$, $i=1,2$. Consumers' utility functions are:

$$u_i(x)=\int_0^{1/2}\sqrt{x(t)}\,dt+a_i\int_{1/2}^{1}\sqrt{x(t)}\,dt, \qquad i=1,2,$$

for $x\in E_+$, where $a_1=\frac{1}{2}$, and $a_2=2$. The economy satisfies conditions A1, A2, A3′ and A5.

Let the price $p\in ca[0,1]$ be the Lebesgue measure. We claim that p is arbitrage-free for both consumers. Consider consumer 1. Since

$$u(x)\leqq\int_0^1\sqrt{x(t)}\,dt\leqq\left(\int_0^1 x(t)\,dt\right)^{1/2}=(px)^{1/2}$$

for every $x\in E_+$, we see that $\lim px_n=+\infty$ whenever $\lim u(e+x_n)=\bar{u}=+\infty$. Therefore p is arbitrage-free for consumer 1. By the same argument, p is arbitrage-free for consumer

2. Therefore, the economy is arbitrage-free, and by Proposition 4, the utility set is bounded. However, the utility set is not closed (see Aliprantis, Brown and Burkinshaw (1989), p. 130). Furthermore, there is no equilibrium in this economy.

It is worth pointing out that the arbitrage-free price system p in this example is not viable, i.e. none of the consumers has an optimal demand at p in the commodity space $C[0, 1]$.

We conclude this section with another example. The purpose of it is to underscore our claim that the concept of a free lunch is inadequate for equilibrium analysis of infinite markets. This example shows an economy in which there is a price system which admits no free lunch for every consumer, but there is no equilibrium.

Example 4. We extend Example 2 by adding one more consumer. We have $E=\ell_\infty$ and $E'=\ell_1$. Consumer 1 has consumption set $X_1=\ell_\infty^+$, initial endowment $e_1=(1, 0, 0, \ldots)$, and utility function $u_1(x)=\sum_{n=1}^{\infty}\delta^n(x_n)^{\frac{1}{2}}$, where $x=(x_1, x_2, \ldots)$, and $0<\delta<1$. Consumer 2 has consumption set $X_2=\ell_\infty$, initial endowment $e_2=(1, 1, 1, \ldots)$, and utility function $u_2(x)=\sum_{n=1}^{\infty}\delta^{4n}x_n$. Let the price system $p=(p_1, p_2, \ldots)\in\ell_1$ be given by $p_n=\delta^{4n}$. Note that $u_2(x)=px$. Clearly p admits no free lunch for both consumer 1 (as argued in Example 2), and consumer 2. We claim that there is no equilibrium. One can easily show that p is the only viable price for consumer 2, and hence the only candidate for an equilibrium price. However, as shown in Example 2, p is not arbitrage-free for consumer 1, and, by Proposition 3, not viable for consumer 1. Therefore there is no equilibrium. One can also show that p is the only arbitrage-free price for consumer 2. Since p is not arbitrage-free for consumer 1, the economy is not arbitrage-free.

6. EXAMPLE: SECURITIES MARKET MODEL

In this section we present an example which shall illustrate that the results of the preceding sections are suitable for an application to financial markets. The example is a version of the securities market model of Hart (1974) with infinitely many securities. We shall show that the economy in this example is arbitrage-free if investors have homogeneous expectations of security payoffs, and that it satisfies conditions A1, A2, A3′ and A5. Our existence result (Theorem 3) will then be applied to conclude that a sufficient condition for the existence of an equilibrium is the closedness of the utility set. A remarkable finding of this section is that the condition A5 of the non-empty interior of preferred sets holds in an interesting class of models of financial markets.

Let there be m investors and a countably infinite collection of securities indexed by $n=1, 2, \ldots$. A typical portfolio of securities is $x=(x_1, x_2, \ldots)$ with x_n being the number of shares of security n. We shall require that $\sum_{n=1}^{\infty}|x_n|<\infty$, i.e. that the total number of shares (short or long) is finite. Thus the portfolio space is ℓ_1 equipped with its norm topology. Security price space is ℓ_∞—the norm dual of ℓ_1—and so $p=(p_1, p_2, \ldots)\in\ell_\infty$ is a list of prices of all securities with $\sup_n|p_n|<\infty$.

Security payoffs are described as follows: Let $(\Omega, \mathscr{F}, P)$ be a probability space (state space). The payoff of security n is $r_n\in\mathscr{L}_\infty(\Omega, \mathscr{F}, P)$, i.e. an (essentially) bounded random variable r_n. To simplify notation we shall denote $\mathscr{L}_\infty(\Omega, \mathscr{F}, P)$ by $\mathscr{L}_\infty$. We assume that there is a riskless security, say security 1, with $r_1(\omega)=1$ for each $\omega\in\Omega$. Furthermore, we assume that for all n, $r_n\in C$ for some (sup norm) bounded set $C\subset\mathscr{L}_\infty^+$. Investors have homogeneous expectations and they all expect security payoffs to be as described above.

Investors never plan to have negative end-of-period wealth. The feasible portfolio set of investor i is $\Gamma = \{x \in \ell_1 : \sum_{n=1}^{\infty} x_n r_n \geq 0\}$, where the inequality in the definition of Γ is with respect to the order of $\mathscr{L}_\infty$, i.e., it holds with P-probability one. Initial portfolio of investor i is $\bar{x}^i \in \Gamma$. Let $\bar{x} = \sum_{i=1}^{m} \bar{x}^i$ denote the outstanding portfolio of securities.

Each investor has a von Neumann-Morgenstern utility function of wealth $u_i : R_+ \to R$ and evaluates a portfolio according to the expected utility of its payoff. We assume that u_i is concave, continuous, and strictly increasing. Let $V_i : \Gamma \to R$ be the indirect utility of a portfolio, i.e. $V_i(x) = Eu_i(\sum_{n=1}^{\infty} x_n r_n)$, where the expected value is taken with respect to the probability measure P. Clearly, feasible portfolio set Γ and indirect utility function V_i satisfy conditions A1, A2 (with $v_i = (1, 0, \ldots)$), and A3′.

The securities market economy described above is an example of an abstract exchange economy of Section 2. Accordingly, an equilibrium consists of a portfolio allocation $(x^1, x^2, \ldots, x^m)$, and a price system $p \in \ell_\infty$ such that $\sum_{i=1}^{m} x^i = \bar{x}$ and each portfolio x^i maximizes $V_i(x)$ over $x \in \Gamma$ subject to the constraint $px \leq p\bar{x}^i$.

A free lunch for investor i (in the sense of Definition 1) is a portfolio x such that $\sum_{n=1}^{\infty} x_n r_n \geq 0$, $P(\sum_{n=1}^{\infty} x_n r_n > 0) > 0$, and $px \leq 0$. This is the standard concept of finance. A sequential arbitrage opportunity for investor i is a sequence of portfolios $\{x^k\} \subset \ell_1$, such that $V_i(\bar{x}^i + x^k) \to \bar{V}_i$, and $\lim px^k \leq 0$, where $\bar{V}_i = \sup_{x \in \Gamma} V_i(x)$.

Let us consider the price system $\bar{p} \in \ell_\infty$ given by $\bar{p}_n = Er_n$, $n = 0, 1, \ldots$. By Jensen's inequality $V_i(x) = Eu_i(\sum x_n r_n) \leq u_i(\sum x_n Er_n) = u_i(\bar{p}x)$. Consequently, if $V_i(\bar{x}^i + x^k) \to \bar{V}_i$ for a sequence $\{x^k\} \subset \ell_1$, then $\bar{V}_i \leq u_i(\bar{p}\bar{x}^i + \lim \bar{p}x^k)$. Therefore, $\bar{p}x^k \to +\infty$, and $\bar{p}$ is arbitrage-free for every investor. The securities market economy is arbitrage-free.

We claim that condition A5 of the non-empty interior of preferred sets is satisfied. To this end let us consider the set Γ of portfolios with non-negative payoffs, and a portfolio $v = (1, 0, \ldots)$ consisting of the riskless security only. We will show that $v \in \operatorname{int} \Gamma$, where int denotes norm interior in ℓ_1. Let $K = \sup_n \|r_n\|_\infty$. By our assumptions $1 \leq K < \infty$. Let $z \in \ell_1$ be such that $\|v - z\|_1 < 1/K$. It suffices to show that $z \in \Gamma$. We have $|1 - \sum_{n=1}^{\infty} z_n r_n(\omega)| = |\sum_{n=1}^{\infty} (v_n - z_n) r_n(\omega)| \leq K \cdot \|v - z\|_1 < 1$ holds with P-probability one. Therefore $\sum_{n=1}^{\infty} z_n r_n(\omega) > 0$, and $z \in \Gamma$. Since $x + \Gamma \subset P_i(x)$ for every $x \in X_i$, $P_i(x)$ has non-empty interior.

Summing up, this version of the securities market model with infinitely many securities satisfies conditions A1, A2, A3′, A5, and is arbitrage-free. If the utility set is closed,[2] then by Theorem 3 there exists a quasi-equilibrium. A quasi-equilibrium is an equilibrium if investors' initial portfolios are positive, i.e. $\bar{x}_i > 0$.

APPENDIX

Proof of Theorem 1. There are two cases. First, the trivial case where the initial allocation $(e_1, \ldots, e_m)$ is weakly Pareto optimal. Let $G = \sum_{i=1}^{m} \prod_i (e_i) - e$, where $\prod_i(x) = \{x' \in X_i : u_i(x') > u_i(x)\}$ for $x \in X_i$. Clearly G is convex and has non-empty interior by assumptions A3 and A5. Weak Pareto optimality of the allocation $(e_1, \ldots, e_m)$ implies that $0 \notin G$. Hence by the standard separation theorem, there exists a price system $p \in E'$, $p \neq 0$ such that $py \geq 0$ for every $y \in G$. Since each utility function u_i is locally non-satiated, it follows by a standard argument that p supports e_i for every $i = 1, \ldots, m$. Consequently, p is quasi-equilibrium price system.

2. One important case in which the utility set is closed is when the market span (i.e. the set of payoffs of all portfolios) is a closed subspace of the payoff space, and the payoff space has weakly compact order intervals (a property satisfied by most of infinite-dimensional commodity spaces, see Mas-Colell and Zame (1991), or Aliprantis, Brown and Burkinshaw (1989)). Another case is when the set of Pareto optimal portfolio allocations resides in a finite-dimensional subspace of the space of portfolio allocations (as in Connor's (1984) Equilibrium APT).

For the case where $(e_1, \ldots, e_m)$ is not weakly Pareto optimal, we shall follow the Negishi's approach to existence of equilibria. This argument requires a series of lemmas. For convenience, we shall assume throughout the proof that $u_i(e_i)=0$ for every i.

Lemma 3. *The utility set U satisfies the following property: there is some $r>0$ such that $0 \leqq z \in R^m$ and $\|z\| \leqq r$ imply $z \in U$.*

Proof of Lemma 3. Since $(e_1, \ldots, e_m)$ is not weakly Pareto optimal, there is an attainable allocation $(x_i)_{i=1}^m \in A$ such that $u_i(x_i) > u_i(e_i)=0$. We set $r=\min\{u_i(x_i): i=1, \ldots, m\}$. ||

Let $\hat{U}=\{u \in R^m: u_i \leqq u_i(x_i),\ i=1, \ldots, m, \text{ for some } (x_i)_{i=1}^m \in A\}$, then $U=\hat{U} \cap R_+^m$. By assumption (A1), U is compact. Let $\delta U = bd\hat{U} \cap R_+^m$, where $bd\hat{U}$ denotes the boundary of the set $\hat{U}$ in R^m.

Lemma 4. *δU is homeomorphic to the simplex Δ of R^m.*

Proof of Lemma 4. The homeomorphism $\phi: \Delta \to \delta U$ is given by $\phi(s)=\rho(s)s$, where $s \in \Delta$ and $\rho(s)= \sup\{\alpha>0: \alpha s \in U\}$. A proof that ϕ is a homeomorphism can be found in Moore (1975). Lemma 3 guarantees that the maps are well-defined. ||

For each $s \in \Delta$, let $(x_i^s)_{i=1}^m \in A$ be an attainable allocation such that $u_i(x_i^s) \geqq \phi_i(s)$, for each i. Allocation $(x_i^s)_{i=1}^m$ is weakly Pareto optimal.

Lemma 5. *There exists V—an open, symmetric neighbourhood of 0 in E such that for every $s \in \Delta$ there exists an attainable allocation $(x_i)_{i=1}^m \in A$ such that $u_i(x_i+z_i) > \phi_i(s)$ for every $z_i \in v_i + V$, and every i.*

Proof of Lemma 5. By Assumptions A2 and A5, for every $s \in \Delta$ there exists V_i^s—an open symmetric neighbourhood of 0 such that $u_i(x_i^s+z_i) > u_i(x_i^s)+\varepsilon_i^s$ for every $z_i \in v_i + V_i^s$, for some $\varepsilon_i^s>0$ (e.g. $\varepsilon_i^s = \frac{1}{2}(u_i(x_i^s+v_i)-u_i(x_i^s))$. Since $u_i(x_i^s) \geqq \phi_i(s)$ and ϕ is continuous, there is an open neighbourhood U_s of s in Δ such that $u_i(x_i^s+z_i) > \phi_i(t)$ holds for every $t \in U_s$, every i, and every $z_i \in v_i + V_i^s$. The family $\{U_s\}_{s \in \Delta}$ is an open covering of Δ which is compact. Therefore there exists a finite sub-covering $U_{s_1}, U_{s_2}, \ldots, U_{s_k}$. We have that for every $s \in \Delta$ there exists s_j for $1 \leqq j \leqq k$ such that $u_i(x_i^{s_j}+z_i) > \phi_i(s)$ holds for every $z_i \in v_i + V_i^{s_j}$, for every i. Taking $V=\bigcap_{i=1}^m \bigcap_{j=1}^k V_i^{s_j}$ we conclude the proof. ||

Let $v=\sum_{i=1}^m v_i$. We define a price set $P=\{p \in E': pv=1 \text{ and } |pw| \leqq 1 \text{ for } w \in V\}$. By Alaoglu's Theorem, the set P is compact in the *weak** topology (denoted by w^*) of E'. Following Moore (1975), we define for each $s \in \Delta$,

$$P(s)=\{p \in P: \text{for every allocation } (x_i)_{i=1}^m, \text{ if } u_i(x_i) \geqq \phi_i(s),\ i=1, \ldots, m,$$
$$\text{then } pz \geqq 0 \text{ for } z=\textstyle\sum_{i=1}^m x_i - e\}.$$

Lemma 6. *$P(s)$ is non-empty and convex for $s \in \Delta$.*

Proof of Lemma 6. Let $\prod_i=\{x \in X_i: u_i(x) > \phi_i(s)\}$, and $G=\sum_{i=1}^m \prod_i - e$. It follows from Lemma 5 that there is an allocation $(x_i)_{i=1}^m \in A$ such that $u_i(x_i+z_i) > \phi_i(s)$ for every $z_i \in v_i + V$. Since $\sum_{i=1}^m (x_i+z_i) - e = \sum_{i=1}^m z_i$, we have $\sum_{i=1}^m z_i \in G$ for every $z_i \in v_i + V$. Consequently $v + V \subset G$, and G has non-empty interior. We claim that $0 \notin G$. Indeed, $0 \in G$ contradicts $\phi(s) \in \delta U$. By a separation theorem, there is $p \neq 0$ which separates G from 0. Clearly $pz \geqq 0$ for every z as in the definition of $P(s)$. It remains to show that p can be normalized so that $p \in P$. This is done in the following way: We have shown that $v \in G$ and $v + V \subset G$. Therefore $pv \geqq 0$ and $p(v+w) \geqq 0$ for $w \in V$. We claim that $pv>0$. Otherwise $pv=0$ and $0 \leqq p(v \pm w) = \pm pw$, since $w \in V$ implies $v \pm w \in v + V$. Consequently, $pw=0$ for every $w \in V$ which implies $p=0$, a contradiction. We normalize p so that $pv=1$. Then, we also have $-1 \leqq pw \leqq 1$, i.e. $|pw| \leqq 1$. The price system p normalized in the above manner belongs to P, and also to $P(s)$. ||

Lemma 7. *For every $p \in P(s)$, and every i, if $x \in X_i$, and $u_i(x) \geqq \phi_i(s)$, then $px \geqq px_i^s$. In particular, p supports $(x_i^s)_{i=1}^m$.*

Proof of Lemma 7. Let for some i_0, $x \in X_{i_0}$ and $u_{i_0}(x) \geqq \phi_{i_0}(s)$. Consider an allocation $(x_i)_{i=1}^m$ defined by $x_i = x_i^s$ for $i \neq i_0$ and $x_{i_0}=x$. For every i, $u_i(x_i) \geqq \phi_i(s)$. Let $z=\sum_{i=1}^m x_i - e$. We have $pz \geqq 0$ for $p \in P(s)$. However, $z = x_{i_0} - x_{i_0}^s$ and therefore $px_{i_0} \geqq px_{i_0}^s$. ||

Define the following correspondence:

$$\Phi(s)=\{(y_1,\ldots,y_m)\in R^m: y_i=p(e_i-x_i^s) \text{ for every } i, \text{ for some } p\in P(s)\}$$

Lemma 8.
(i) *The range of* Φ *is bounded.*
(ii) Φ *has closed graph, and is convex-valued.*

Proof of Lemma 8. (i) Suppose not, then there are sequences $\{y^n\}\subset R^m$, $\{s_n\}\subset\Delta$, and $\{p_n\}\subset E'$ such that $\|y^n\|\to+\infty$ and $y_i^n=p_n(e_i-x_i^{s_n})$. We have $p_n e=\sum_{i=1}^m p_n x_i^{s_n}$ and $\sum_{i=1}^m y_i^n=0$. But $p_n e$ is bounded because $p_n\in P$, and P is w^*-compact. Moreover, $p_n x_i^{s_n}$ is uniformly bounded above. Indeed, let $\bar{x}_i\in X_i$ be such that $u_i(\bar{x}_i)=\max\{u_i: u\in U\}$. Since p_n supports $x_i^{s_n}$, we have $p_n x_i^{s_n}\leqq p_n\bar{x}_i$, for every n. However, $p_n\bar{x}_i$ is bounded above by the same argument as above. Therefore $p_n x_i^{s_n}$ is uniformly bounded above and below which contradicts $|y_{i_0}^n|\to+\infty$ for some i_0. Hence $y\in\Phi(s)$ implies $|y_i|\leqq\delta$ for some $\delta>0$ and every i and $s\in\Delta$.

(ii) Let $y=\lim y^n$, $s=\lim s_n$, and $y^n\in\Phi(s_n)$. We shall prove that $y\in\Phi(s)$. Since P is w^*-compact, we may assume that there is $p\in P$ such that $p_n\to p$ in w^*-topology. By assumption A2, $u_i(x_i^s+\varepsilon v_i)>u_i(x_i^s)\geqq\phi_i(s)$ for $0<\varepsilon\leqq 1$. Since ϕ is continuous, we have $u_i(x_i^s+\varepsilon v_i)>\phi_i(s_n)$ for n large enough. Applying Lemma 7, we obtain $p_n(x_i^s+\varepsilon v_i)\geqq p_n x_i^{s_n}=p_n e_i-y_i^n$. Passing to the limit, we see that $px_i^s+\varepsilon pv_i\geqq pe_i-y_i$ for every $0<\varepsilon\leqq 1$. This implies, $px_i^s\geqq pe_i-y_i$. Since $\sum_{i=1}^m y_i=0$ and $\sum_{i=1}^m x_i^s=\sum_{i=1}^m e_i$, we obtain $y_i=p(e_i-x_i^s)$.

It remains to show that $p\in P(s)$. Let $(x_i)_{i=1}^m$ be an allocation such that $u_i(x_i)\geqq\phi_i(s)$ for every i, and let $z=\sum_{i=1}^m x_i-e$. By Assumption A2, $u_i(x_i+\varepsilon v_i)>u_i(x_i)$ for every $0<\varepsilon\leqq 1$. By continuity of ϕ, we obtain $u_i(x_i+\varepsilon v_i)>\phi_i(s_n)$ for large n, and every i. Since $p_n\in P(s_n)$, we have $p_n z+\varepsilon p_n v\geqq 0$. Passing to the limit, $pz+\varepsilon pv\geqq 0$ for every ε. Therefore $pz\geqq 0$ and $p\in P(s)$. ||

The rest of the proof of Theorem 1 is a standard application of Kakutani's fixed-point theorem to show that $0\in\Phi(\bar{s})$ for some $\bar{s}\in\Delta$. The details of the argument can be adopted from the proof of Theorem 3.5.12 in Aliprantis, Brown and Burkinshaw (1989). The only point which requires clarification is the proof that if $s_i=0$ for some $s\in\Delta$, then $pe_i-px_i^s\geqq 0$, for $p\in P(s)$. In our case, if $s_i=0$ then $\phi_i(s)=0=u_i(e_i)$. The inequality $pe_i-px_i^s\geqq 0$ follows therefore from Lemma 7.

Clearly if $0\in\Phi(\bar{s})$ then the attainable allocation $(x_i^{\bar{s}})_{i=1}^m$ is a quasi-equilibrium with some price $p\in P(\bar{s})$. ||

Recession (asymptotic) cone of a set.

Let C be a closed and convex subset of E. The recession cone of C is $AC=\{\bar{x}\in E: x+\lambda\bar{x}\in C$ for every $x\in C$ and $\lambda\geqq 0\}$. The following is true:

(1) $AC=\{\bar{x}\in E: \bar{x}+C\subset C\}$,
(2) $AC=\{\bar{x}\in E: x+\lambda\bar{x}\in C$ for every $\lambda\geqq 0\}$ for arbitrary $x\in C$,
(3) $AC=\bigcap_{\lambda>0}\lambda(C-\{x\})$ for $x\in C$, and therefore AC is closed.

Lemma. *For a closed and convex set* C, $AC=\{\bar{x}\in E: \bar{x}=\lim\lambda_n x_n$ *for some sequences* $\{x_n\}\subset C$ *and* $\{\lambda_n\}\subset R_+$ *with* $\lambda_n\to 0\}$.

Proof. Let $\bar{x}\in AC$. Clearly $x+n\bar{x}\in C$ for every n and $\bar{x}=\lim 1/n(x+n\bar{x})$. Conversely, let $\bar{x}=\lim\lambda_n x_n$ for some $\{x_n\}\subset C$ and $\lambda_n\to 0$. For n large enough, $\lambda_n<1$ and, by convexity of C, $\lambda_n x_n+(1-\lambda_n)x\in C$ for every $x\in C$. By closedness, taking limits as $n\to\infty$, we obtain $\bar{x}+x\in C$, i.e. $\bar{x}\in AC$. ||

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